9.5 Composition of Functions

A Composition of Functions

Any function $y = f(x)$ can be imagined as a machine f who changes the input x into the output y:

$$x \to \boxed{f} \to y$$

Let connect now the output of the function machine g as the input of the function machine f corresponding to a different function $z = f(y)$:

$$x \to \boxed{g} \to y \to \boxed{f} \to z$$

Then, the following expressions can be written:

$$y = g(x),\ z = f(y),\ \ z = f(g(x))$$

So, we can replace the machines g and f by a new machine $f \circ g$ who changes the input x directly into the output z:

$$x \to \boxed{f \circ g} \to z$$

So, the relation

$$(f \circ g)(x) = f(g(x))$$

defines the composition of the functions f and g.

Similarly:

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ f)(x) = f(f(x))$$
$$(g \circ g)(x) = g(g(x))$$

Ex 1. Consider the following functions $f(x) = x + 1$, $g(x) = x^2 - 2$, $h(x) = \sqrt{x} + 3$. Find:

a) $(f \circ g)(0)$

b) $(g \circ f)(2)$

c) $(f \circ h)(4)$

d) $(h \circ f)(0)$

e) $(f \circ f)(2)$

f) $(g \circ g)(1)$

g) $(h \circ h)(1)$

i) $(f \circ g \circ h)(1)$

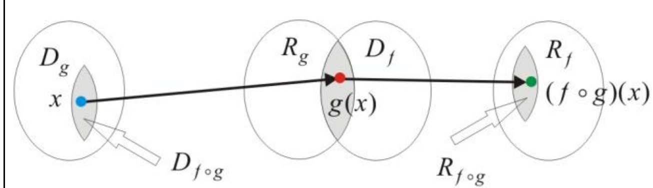
B Domain and Range

The domain of $f \circ g$ is a subset of the domain of g:

$$D_{f \circ g} = \{x \in D_g \mid g(x) \in D_f\}$$

The domain of $f \circ g$ consists of the numbers x in the domain of g such that $g(x)$ is in the domain of f.

The range of $f \circ g$ is a subset of the range of f.



Ex 2. The functions f and g are given by the following mapping diagrams. Find each composition and then find the domain and the range of the composition.

f:
1 → 2
2 → -1
3 → 0

g:
-1 → -3
0 → 2
1 → 3

a) $f \circ g$

b) $g \circ f$

c) $f \circ f$

d) $g \circ g$

Ex 3. Consider the following functions $f(x) = 2x - 1$, $g(x) = x^2$, $h(x) = \sqrt{x-1}$, and $k(x) = \frac{1}{x+1}$. Find the following compositions and for each case state the domain and the range.

a) $(f \circ g)(x)$

b) $(k \circ k)(x)$

c) $(f \circ g \circ h)(x)$

d) $(k \circ g \circ f)(x)$

Ex 4. Prove that:

a) $(f \circ f^{-1})(x) = x$

b) $(f^{-1} \circ f)(x) = x$

Ex 5. Prove that:

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Reading: Nelson Textbook, Pages 545-551
Homework: Nelson Textbook, Page 552: #5aef, 6ace, 7ac, 9, 13, 16

